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Quantum mechanics on Riemannian manifold in Schwinger's quantization approach II

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Abstract. The extended Schwinger quantization procedure is used for constructing quantum mechanics on a manifold with a group structure. The considered manifold M is a homogeneous Riemannian space with the given action of an isometry transformation group. Using the identification of M with the quotient space G/H, where H is the isotropy group of an arbitrary fixed point of M, we show that quantum mechanics on G/H possesses a gauge structure, described by a gauge potential that is the connection 1-form of the principal fiber bundle $G(G/H, H)$. The coordinate representation of quantum mechanics and the procedure for selecting the physical sector of the states are developed.

1 Introduction

The purpose of this paper is to propose a natural development of the method described in our previous paper [2], where we have introduced an extension of Schwinger's quantization procedure [1] in order to consider quantum mechanics on a manifold with a group structure. In [2], this approach has been realized for the case of a homogeneous Riemannian manifold admitting the action of a simply transitive group of isometries.

In this paper we consider a more general type of n-dimensional homogeneous Riemannian manifold M with a p-dimensional group of isometries acting on M transitively (but not simply transitively). In this case part of the isometry transformations form an isotropy group of any point of M, so it is possible to treat this group as a group of local gauge transformations.

This paper is organized as follows. In Sect. 2 we briefly examine the geometry structure of a homogeneous Riemannian manifold. Such a manifold is isomorphic to the quotient space G/H, where H denotes the isotropy group of an arbitrary fixed point of M. In Sect. 3 the operator Lagrangian L describing a free particle in the configuration space G is presented in accordance with [2], where its form has been derived by requiring L to be scalar invariant under a general coordinate transformation on G. The extension of the configuration space from $M \cong G/H$ to G causes the problem of fixing the gauge invariance associated with additional degrees of freedom.

To eliminate unphysical states that appear due to the presence of the gauge degrees of freedom we use the $(m+n)$-decomposition that is usual in theories of Kaluza–Klein type [3]. After introducing the special coordinate system on G that provides this decomposition, the algebra of the commutation relations is constructed in Sect. 4. In Sect. 5 we give the Heisenberg equation of motion describing a free particle on G/H. It turns out that the dynamics on G/H is governed by a Lorentz-type force expressed in terms of the gauge field in the usual way. The gauge potential is the same as the connection 1-form of the principal fiber bundle $G(G/H, H)$. In Sect. 6 the coordinate representation of quantum mechanics on G/H is discussed. A special feature of the theory consists of the emergence of a gauge structure induced by some isotropy group $H \subset G$, and described by the concrete unitary representation of H in the space of states. Different irreducible representations determine unequivalent quantum theories on G/H. The corresponding quantum states are classified by eigenvalues of the Casimir operator of the representation of H.

The conclusions obtained from the model we are considering are in accordance with the concepts introduced in [4], where the method of investigation is somewhat different from ours.

2 Structure of homogeneous Riemannian manifold

A smooth manifold M is called *homogeneous*, if it admits the transitive action of a Lie group G. We assume that $\dim(M) = n$, $\dim(G) = p > n$. This assumption means that the action of G on M is not simply transitive; the case of a simply transitive transformation group (i.e. when $p = n$) has been investigated in our previous paper [2].

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A left action of G on M is determined by the following differentiable map:

$$\begin{aligned} \rho_g : M &\longrightarrow M, \\ q &\longrightarrow \overline{q} := \rho_g(q), \end{aligned} \tag{2.1}$$

where $\rho_g \in \mathrm{Diff}(M)$ satisfies the conditions
(1) $\forall g_{1,2} \in G, \quad \rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 g_2}$;
(2) $\rho_e = \mathrm{id}_M, \quad e \in G$ is the unit element (here id_M denotes an identity map on M).

For a point $q \in M$ the subset $I(q) := \{g \in G : \rho_g q = q\} \subset G$ is a subgroup of G. This group is called the *isotropy group* (or *stabilizer*) of $q \in M$. As the action ρ of G is transitive, all isotropy groups are conjugate, i.e. $\forall \overline{q} = \rho_g q, \quad I(\overline{q}) = \mathrm{aut}_g I(q) := g I(q) g^{-1}$.

Fixing an arbitrary point $q_0 \in M$, we introduce a subgroup $H := I(q_0) \subset G$, $\dim(H) = m$. By means of this subgroup we construct the quotient space $G/H = \{gH : g \in G\}$, $\dim(G/H) = p - m = n$. We denote the element of G/H by $[g]$, where the element $g \in G$ in brackets represents the equivalence class gH.

We define a canonical projection

$$\begin{aligned} p : G &\longrightarrow G/H, \\ q &\longrightarrow p(g) := [g], \end{aligned} \tag{2.2}$$

which determines the structure of the principal fiber bundle $G(G/H, H)$ with the total space G, the base space G/H and the structure group H. The fiber under the point $[g] \in G/H$ is $p^{-1}([g]) = gH \subset G$.

On the other hand, we can naturally define the left transitive action of G on G/H by the following map:

$$\begin{aligned} \rho_g : G/H &\longrightarrow G/H, \\ [g_1] &\longrightarrow \rho_g[g_1] := [g g_1]. \end{aligned} \tag{2.3}$$

The stabilizer of $[e] \in G/H$ under the transformation (2.3) is the subgroup $H \subset G$, because $\forall g \in G$, $[gH] = [g]$. For $[g] \in G/H$ we have $I([g]) = gHg^{-1}$.

Hence, there is a one-to-one correspondence between the points of M and those of G/H. Namely, $q_0 \in M$ and $q = \rho_g q_0$ correspond to $[e] \in G/H$ and $[g] \in G/H$ respectively. We will identify M with G/H throughout this text.

In order to analyze the local properties of the principal fiber bundle $G(G/H, H)$ and the metric structure of G and G/H we introduce local coordinates on G, H and G/H.

Hereafter we utilize notation of indices as follows. The first lot of Latin capital letters $A, B, \ldots = \overline{1,p}$ is used to represent the frame $\{T_A|_e : A = \overline{1,p}\}$ of T_eG (the space of tangent vectors to G at $e \in G$). The final lot of Latin capital letters $M, N \ldots = \overline{1,p}$ is used to mark the local coordinates $\{x^M(g) : M = \overline{1,p}\}$ of $g \in G$.

The structure equation for a (left) Lie algebra Lie(G), which consists of left translations of the elements of T_eG, has the usual form

$$[L_A, L_B] = c^C{}_{AB} L_C, \tag{2.4}$$

where $c^C{}_{AB}$ are structure constants of G, L_A is a left-invariant vector field over G, defined as $L_A|_g := \mathrm{d}_e L_g (T_A|_e)$ (L_g denotes a left translation map, see Appendix A).

As far as H is a Lie subgroup of G, for the corresponding Lie algebras we have a similar inclusion: Lie(H) $\subset$ Lie(G) (subalgebra) and $T_eH \subset T_eG$ (vector subspace). We can choose the basic fields of T_eG in such a way that part of them forms the basis of T_eH. So we can decompose the set $\{T_A : A = \overline{1,p}\}$ as $T_A = (T_a, T_i)$, where $\{T_i : i = \overline{n+1,p}\}$ is the basis of T_eH, marked by small Latin letters $i, j, k \ldots = \overline{n+1,p}$, with the structure equation

$$[L_i, L_j] = c^k{}_{ij} L_k, \quad c^a_{ij} = 0, \tag{2.5}$$

and $\{T_a : a = \overline{1,n}\}$ corresponds to the remaining part of the basic vectors of T_eG. In other words, the index $A = \overline{1,p}$ can be decomposed as $A = (a, i)$.

Due to (2.5) the following system of partial differential equations has n independent solutions:

$$L^M_i \partial_M \varphi(x(g)) = 0. \tag{2.6}$$

These n independent solutions are $\{\varphi^\alpha(x(g)) : \alpha = \overline{1,n}\}$. Here $L = \{L^M_A\}$ denotes the matrix of left translations on G (see Appendix A for its properties).

Now we introduce a new coordinate system on G by means of the following transformation:

$$\begin{cases} \overline{x}^\alpha = \varphi^\alpha(x(g)), \\ \overline{x}^\mu = x^\mu. \end{cases} \tag{2.7}$$

Hereafter $\{x^M\}$ are assumed to refer to the new coordinate system on G. Therefore, the coordinate index $M = \overline{1,p}$ is decomposed like the one for the group, namely $M = (\alpha, \mu)$, where $\alpha, \beta \ldots = \overline{1,n}$ and $\mu, \nu \ldots = \overline{n+1,p}$. The meaning of $\{x^\alpha\}$ and $\{x^\mu\}$ will be determined later.

In new coordinates the matrix of left translations on G receives the form

$$L = \{L^M_A\} = \begin{pmatrix} L^\alpha_a & L^\mu_a \\ 0 & L^\mu_i \end{pmatrix}, \quad L^\alpha_i = 0. \tag{2.8}$$

If $\det(L) = \det(L^\alpha_a)\det(L^\mu_i) \neq 0$, the matrix (2.8) has the inverse

$$L^{-1} := \overline{L} = \{\overline{L}^A_M\} = \begin{pmatrix} \overline{L}^a_\alpha & \overline{L}^i_\alpha \\ 0 & \overline{L}^i_\alpha \end{pmatrix}, \quad \overline{L}^a_\mu = 0, \tag{2.9}$$

where

$$\overline{L}^a_\alpha = (L^{-1})^\alpha_a, \quad \overline{L}^i_\mu = (L^{-1})^\mu_i, \quad \overline{L}^i_\alpha = -\overline{L}^a_\alpha L^\mu_a \overline{L}^i_\mu. \tag{2.10}$$

The matrices $\{L^a_\alpha\}$, $\{L^i_\mu\}$ and their inverses satisfy the equations following from (2.4) and the Maurer–Cartan equation taking into account the fact that $c^a{}_{ij} = 0$.

As far as a left Lie algebra commutes with a right one we can generally write $R^M_A \partial_M L^N_B = L^M_B \partial_M R^N_A$ and, in particular,

$$\begin{cases} \partial_\mu R^\alpha_A = 0, \\ \partial_\alpha R^\beta_A = R^M_A \overline{L}^a_\alpha \partial_M L^\beta_a. \end{cases} \tag{2.11}$$

As to the other relations like (2.11): their explicit forms are not important in the following consideration.

The set of coordinates $\{x^\alpha : \alpha = \overline{1,n}\}$ in the decomposition $x^M = (x^\alpha, x^\mu)$ is independent of the point at the orbit: $\forall h \in H$, $x^\alpha(gh) = x^\alpha(g)$. To prove this statement one can rewrite $x^M(gh)$ as a Taylor expansion

$$x^M(gh) = x^M(g) + x^M(h) + \ldots$$

taking into account $L_i^\alpha = 0$.

Therefore, we can write the local form of the projection map as

$$\begin{aligned} p: \qquad G \qquad &\longrightarrow G/H, \\ \{x^\alpha(g), x^\mu(g)\} &\longrightarrow \{x^\alpha(g)\}. \end{aligned} \tag{2.12}$$

Hence, the local coordinate system on G/H can be defined by

$$x^\alpha([g]) := x^\alpha(g). \tag{2.13}$$

According to (2.2) and (2.12), the action of G on the quotient space can be represented as follows:

$$x^\alpha(\rho_g[g_1]) = x^\alpha([gg_1]) = x^\alpha(gg_1). \tag{2.14}$$

Let $g_A(\tau) \in G$, $g_A(0) = e$ be the integral curve of the basic vector field:

$$L_A\big|_e = L_A^M(e)\,\partial_M\big|_e = T_A\big|_e \in T_eG. \tag{2.15}$$

Then the corresponding vector field over G/H, induced by the action of G on G/H, has the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} x^\alpha(g_A(\tau)g) \\ &= \frac{\partial x^\alpha(g_A(\tau)g)}{\partial x^M(g_A(\tau))}\bigg|_{g_A=e} \cdot \frac{\mathrm{d}x^M(g_A(\tau))}{\mathrm{d}\tau}\bigg|_{\tau=0} \\ &= R_M^\alpha(g)\delta_A^M = R_A^\alpha(g). \end{aligned} \tag{2.16}$$

In particular,

$$\forall h \in H, \quad R_A^\alpha(h) = R_A^\alpha(e) = \delta_A^\alpha$$

determines the transformations of $I([e])$ on G/H, as it must.

In the special coordinate system, defined in (2.7), using the diagonal form of the metric of T_eG (see Appendix A)

$$\{\eta_{AB}\} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ij} \end{pmatrix}, \tag{2.17}$$

we find the following form for the left-invariant metric of G:

$$\eta_{MN} := \eta_{AB}\overline{L}_M^A\overline{L}_N^B = g_{ab}\overline{L}_M^a\overline{L}_N^b + g_{ij}\overline{L}_M^i\overline{L}_N^j; \tag{2.18}$$

$$\eta^{MN} := \eta^{AB}L_A^M L_B^N = g^{ab}L_a^M L_b^N + g^{ij}L_i^M L_j^N. \tag{2.19}$$

Using the definition of the connection on the principal fiber bundle $G(G/H, H)$ (see Appendix B)

$$A_\alpha^\mu := \overline{L}_\alpha^i L_i^\mu = -\overline{L}_\alpha^a L_a^\mu, \tag{2.20}$$

the metric tensor (2.18) and its inverse (2.19) receive the form

$$\begin{aligned} \{\eta_{MN}\} &= \begin{pmatrix} \eta_{\alpha\beta} & \eta_{\mu\beta} \\ \eta_{\alpha\nu} & \eta_{\mu\nu} \end{pmatrix} \\ &= \begin{pmatrix} g_{\alpha\beta} + g_{\rho\sigma}A_\alpha^\rho A_\beta^\sigma & g_{\mu\rho}A^{\rho\beta}, \\ A_\alpha^\rho g_{\rho\nu} & g_{\mu\nu} \end{pmatrix}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \{\eta^{MN}\} &= \begin{pmatrix} \eta^{\alpha\beta} & \eta^{\mu\beta} \\ \eta^{\alpha\nu} & \eta^{\mu\nu} \end{pmatrix} \\ &= \begin{pmatrix} g^{\alpha\beta} & -g^{\gamma\beta}A_\gamma^\mu \\ -g^{\alpha\gamma}A_\gamma^\nu & g^{\mu\nu} + g^{\gamma\delta}A_\gamma^\mu A_\delta^\nu \end{pmatrix}, \end{aligned} \tag{2.22}$$

where we denote the metric tensors of G/H and H by

$$g_{\alpha\beta} = g_{ab}\overline{L}_\alpha^a\overline{L}_\beta^b, \quad g^{\alpha\beta} = g^{ab}L_a^\alpha L_b^\beta \tag{2.23}$$

and

$$g_{\mu\nu} = g_{ij}\overline{L}_\mu^i\overline{L}_\nu^j, \quad g^{\mu\nu} = g^{ij}L_i^\mu L_j^\nu, \tag{2.24}$$

respectively.

Note that a similar form of the metric tensor (2.21) and (2.22) appears in the Kaluza–Klein scheme [3].

The right-invariant vector fields $R_A = R_A^M\partial_M$ are the Killing vectors for the left-invariant metric η_{MN} of G, as follows from the definition of $\{R_A^M\}$.

Using (2.21) we can rewrite the Killing equation for $\{\eta_{MN}\}$ in the following form:

$$-R_A^\gamma\partial_\gamma g^{\alpha\beta} + g^{\alpha\gamma}\partial_\gamma R_A^\beta + g^{\beta\gamma}\partial_\gamma R_A^\alpha = R^\mu\partial_\mu g^{\alpha\beta}. \tag{2.25}$$

On the other hand, we can define the metric of G/H from the metric $\{\eta_{MN}\}$ of G, requiring the map dp to be an isometry.

A direct calculation shows that the metric of G/H is described by $\{g_{\alpha\beta}\}$. If we treat G/H as a homogeneous Riemannian manifold, we have to conclude from (2.25) that $\partial_\mu g^{\alpha\beta} = 0$. This condition means that the Lie variation of η_{MN} for left-invariant vector fields $L_A = L_A^M\partial_M$ vanishes, i.e. $\{\eta_{MN}\}$ is invariant. This is possible if and only if G is a semisimple group. In this case the metric $\{\eta_{AB}\}$ of T_eG can be identified with an adjoint-invariant Cartan–Killing form.

It is useful to write down here the explicit form of the Killing equations $\delta\eta_{MN} = 0$ for left-invariant vector fields in the special coordinate system

$$\begin{aligned} &\partial_\mu g_{\alpha\beta} = 0, \\ &L_i^\mu\partial_\mu A_\alpha^\nu = A_\alpha^\mu\partial_\mu L_i^\nu - \partial_\alpha L_i^N, \\ &L_i^\sigma\partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu L_i^\sigma + g_{\nu\sigma}\partial_\mu L_i^\sigma = 0. \end{aligned} \tag{2.26}$$

Hence, we have shown that the homogeneous Riemannian manifold M can be identified with the quotient space G/H, where G is the isometry group of M and $H = I(q_0)$ is a stabilizer of an arbitrary fixed point $q_0 \in M$. The general geometric analysis of this problem is presented in [5].

3 Quantum Lagrangian for free particle in homogeneous Riemannian space

Now we consider quantum mechanics for a free particle in a homogeneous Riemannian space M, $\dim(M) = n$ with a given action of the group G of isometries, and $\dim(G) = p > n$. As has been shown in the previous section, the configuration space M can be identified with the quotient space G/H, where H, $\dim(H) = m = p - n$, denotes the stabilizer of an arbitrary fixed point of M.

In the construction of quantum theory, based on Schwinger's quantization approach, a key role is played by the realization of the Lie algebra Lie(G) of G induced by the realization of G on M. The set of independent Killing vectors, associated with the basis of Lie(G), possesses the properties of permissible variations $\{\delta q^\mu\}$, where $\{q^\mu\}$ denote the set of coordinate operators describing the position of a particle.

In the case of the homogeneous Riemannian manifold G/H the number $p = \dim(G)$ of independent Killing vectors is higher than the dimension n of the manifold G/H. Some of these vectors form the representation of the isotropy group $I([g])$, $\dim(I([g])) = m = p - n$, that depends on the choice of a point $[g] \in G/H$. The transformations of $I([g])$ can be treated as local gauge ones. Therefore, we can divide the independent Killing vectors at $[g] \in G/H$ into two sets. The first one generates nontrivial transformations on G/H, the second one realizes the action of the stabilizer subgroup $I([g]) \subset G$ on M (the group of local gauge transformations).

Making an attempt to realize Schwinger's quantization procedure immediately as in [2], one observes the same difficulties as in the usual gauge models. Here, as in any theory with first-class constraints, there is the problem of fixing the gauge degrees of freedom, which number in our model equals $m = \dim(G) - \dim(G/H)$.

This procedure is performed in the present model by means of introducing a new configuration space G (the space of the isometry group). The local coordinate system in G is described by

$$x^M = \{x^M(g) : x^M(g) = \{x^\alpha([g]), x^\mu(g)\}, g \in G\},$$
$$M = \overline{1,p}, \quad \alpha = \overline{1,n}, \quad \mu = \overline{n+1,p}, \tag{3.1}$$

where $\{x^\mu()\}$ are coordinates in the orbit gH.

The metric $\{\eta_{MN}(g) : M, N = \overline{1,p}\}$ has been defined in the previous section by the formulae (2.21) and (2.22).

Of course, the enlargement of the number of degrees of freedom from $n = \dim(G/H)$ to $p = \dim(G)$ brings about the appearance of unphysical quantum states. The procedure for their elimination will be presented later.

The quantum Lagrangian describing a free particle in the new configuration space G has the following form:

$$L_G := \frac{1}{2}\dot{x}^M \eta_{MN}(g)\dot{x}^N - U_q(g); \tag{3.2}$$

it has been introduced in [2]. Here $U_q(g)$ denotes a so-called "quantum potential". Its role consists in providing the scalar invariance of (3.2) under a general coordinate transformation $x^M \to \overline{x}^M = \overline{x}^M(x)$ on G (note that $[x^M, \dot{x}^N] \neq 0$).

Taking into account the $(m+n)$-decomposition of the metric in the special coordinate system, introduced in the previous section, the Lagrangian (3.2) can be written as

$$L_G = \frac{1}{2}\dot{x}^\alpha g_{\alpha\beta}([g])\dot{x}^\beta + \frac{1}{2}\left(\dot{x}^\mu + \dot{x}^\alpha A^\mu_\alpha(g)\right)$$
$$\times\, g_{\mu\nu}(g)\left(\dot{x}^\nu + A^\nu_\beta(g)\dot{x}^\beta\right), \tag{3.3}$$

where the first term corresponds to the kinetic energy of a particle on G/H, while the appearance of the second term is caused by the extension of the physical configuration space from G/H to G.

Since G acts on itself simply transitively, the method of constructing quantum theory based on the Lagrangian (3.2) (or (3.3)) coincides with that of the one developed in [2]. Note that due to the semisimplicity of G, there are two equivalent sets of Killing vectors $\{v^M_A(g)\}$, that correspond to the matrices of left and right translations on G, i.e. $\{L^M_A(g)\}$ and $\{R^M_A(g)\}$ respectively.

In accordance with [2] the permissible variations of the coordinate operators $\{x^M\}$ can be written in the following form:

$$\delta x^M = \varepsilon^A v^M_A(g),$$
$$G = \varepsilon^A v^M_A \circ p_M, \quad p_M := \eta_{MN} \circ \dot{x}^N. \tag{3.4}$$

Here ε^A is an infinitesimal c-number parameter of a coordinate transformation on G.

4 Algebra of commutation relations

Constructing the algebra of the commutation relations for operators describing the quantum mechanics of the particle on G/H, we will use the fact that the algebra to be found is contained in the wider one associated with quantum mechanics on G.

At first, we consider the right isometries of the metric $\{\eta_{MN}\}$. In this case, in accordance with the results of the previous sections, the Killing vectors coincide with left-invariant vector fields $\{L^\mu_i(g)\, \partial_\mu|_g\}$, which determine the generator of right translations as follows:

$$G_i = L^M_i \circ p_\mu,$$
$$p_\mu := \eta_{\mu N} \circ \dot{x}^N = g_{\mu\nu} \circ (\dot{x}^\nu + A^\nu_\alpha \circ \dot{x}^\alpha). \tag{4.1}$$

In this case

$$\delta_i x^\alpha = 0 = \frac{1}{\mathrm{i}\hbar}[x^\alpha, L^\mu_i \circ p_\mu],$$
$$\delta_i x^\mu = L^\mu_i = \frac{1}{\mathrm{i}\hbar}[x^\mu, L^\mu_i \circ p_\mu]. \tag{4.2}$$

Since $[x^M, L^\mu_i] = 0$ and $\det(L^\mu_i) \neq 0$ we can conclude from (4.2) that

$$\left[x^M, p_\mu\right] = \mathrm{i}\hbar\delta^M_\mu. \tag{4.3}$$

For the case of an arbitrary function $f(\{x^M\})$ depending on the coordinates $\{x^M\}$ we have

$$\delta_i f = L^M_i \partial_M f = L^\mu_i \partial_i f = \frac{1}{\mathrm{i}\hbar}\left[f, L^\nu_i \circ p_\nu\right]; \tag{4.4}$$

then

$$[f, p_\mu] = \mathrm{i}\hbar\partial_\mu f. \tag{4.5}$$

Taking into account the transformation law of p_μ under the transformation $x^M \to x^M + \delta x^M$ one can easily find

$$[p_\mu, p_\nu] = 0. \tag{4.6}$$

The commutation relations developed in this way determine the quantum mechanics on the orbit gH.

Using the commutation relations (4.3)–(4.6) and the structure equation for $\{L^\mu_i\}$, one can directly prove that

$$[p_i, p_j] = -\mathrm{i}\hbar c^k{}_{ij} p_k, \tag{4.7}$$

where $p_i := L^\mu_i \circ p_\mu$.

Further we consider the left isometries of the metric $\{\eta_{MN}\}$ which are described by the set of Killing vectors $\{R^M_A \partial_M\}$. The generator of these transformations has the form

$$G_A = R^M_A \circ p_M = R^\alpha_A \circ \pi_\alpha + (R^\mu_A + A^\mu_\alpha R^\alpha_A) \circ p_\mu, \tag{4.8}$$

where we denote

$$\begin{aligned} p_\alpha &= \eta_{\alpha M} \circ \dot{x}^M = \pi_\alpha + A^\mu_\alpha \circ p_\mu, \\ \pi_\alpha &= g_{\alpha\beta} \circ \dot{x}^\beta, \\ p_\mu &= g_{\mu\nu} \circ (\dot{x}^\nu + A^\nu_\alpha \circ \dot{x}^\alpha). \end{aligned} \tag{4.9}$$

The vector $\{\pi_\alpha\}$, rewritten as

$$\pi_\alpha = p_\alpha - A^i_\alpha \circ p_i, \quad A^i_\mu = \overline{L}^i_\mu A^\mu_\alpha = \overline{L}^i_\alpha, \tag{4.10}$$

has the meaning of the momentum operator of a free particle on G/H. As far as $\{R^\alpha_A \partial_\alpha\}$ are Killing vectors of G/H, the first term in (4.8) coincides with the generator of the isometry transformations of the metric of G/H, while the second one is caused by the extention of the configuration space from G/H to G.

To define the operator properties of π_α, we consider the variation of an arbitrary function $f(x)$ of only the $\{x^M\}$'s:

$$\delta_A f = R^M_A \partial_M f = \frac{1}{\mathrm{i}\hbar}[f, G_A]. \tag{4.11}$$

Substituting the explicit form of the generator G into (4.11) one can rewrite (4.11) as

$$\begin{aligned} R^\alpha_M \circ [f, \pi_\alpha] &= \mathrm{i}\hbar R^\alpha_A \left(\partial_\alpha - A^i_\alpha L_i\right) f, \\ L_i &= L^\mu_i \partial_i. \end{aligned} \tag{4.12}$$

Using the fact that $\{\overline{R}^A_M\}$ is the inverse of $\{R^\mu_A\}$ we can contract (4.12) with $\overline{R}^A_M$; then

$$[f, \pi_\alpha] = \mathrm{i}\hbar D_\alpha f, \quad D_\alpha := \partial_\alpha - A^i_\alpha L_i. \tag{4.13}$$

The commutator of the new derivatives D_α acts on the scalar function $f(\{x^M\})$ as

$$[D_\alpha, D_\beta] f = -F^i{}_{\alpha\beta} L_i, \tag{4.14}$$

where

$$F^i{}_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha + c^i{}_{jk} A^j_\alpha A^k_\beta. \tag{4.15}$$

A direct calculation leads to

$$[\pi_\alpha, \pi_\beta] = \mathrm{i}\hbar F^i{}_{\alpha\beta} \circ p_i. \tag{4.16}$$

The object $\{F^i{}_{\alpha\beta}\}$ in (4.14)–(4.16) can be treated as a strength tensor of the gauge field A^i_α on G/H. The corresponding gauge group is the isotropy group $H \subset G$.

5 Equations of motion and Hamiltonian

Using the same procedure as developed in [2] one can construct the Hamiltonian for the system in the configuration space G, expressed in terms of the momentum operators $p_M = \eta_{MN} \circ \dot{x}^N$. The Hamiltonian is completely determined by the initial Lagrangian:

$$\begin{aligned} H_G &= \frac{1}{2}\left(p_M - \frac{\mathrm{i}\hbar}{2}\Gamma_M\right)\eta^{MN}\left(p_N + \frac{\mathrm{i}\hbar}{2}\Gamma_N\right) \\ &= \frac{1}{2} p_M \eta^{MN} p_N + V_G(x), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} V_G &= \frac{\hbar^2}{4}\left(\partial_M \Gamma^M + \frac{1}{2}\Gamma_M \Gamma^M\right), \\ \Gamma_M &= \Gamma^N{}_{MN} = \frac{1}{2\det(\eta_{MN})}\partial_M \det(\eta_{MN}). \end{aligned} \tag{5.2}$$

Here $\Gamma^M{}_{N_1 N_2}$ denotes the Christoffel symbol constructed with the metric $\{\eta_{MN}\}$.

Using the properties of the $(m+n)$-decomposition one can rewrite (5.1) as

$$H_G = H_{G/H} + H_{\mathrm{orb}}, \tag{5.3}$$

where

$$\begin{aligned} H_{G/H} &= \frac{1}{2}\left(\pi_\alpha - \frac{\mathrm{i}\hbar}{2}\Gamma_\alpha\right) g^{\alpha\beta}\left(\pi_\beta + \frac{\mathrm{i}\hbar}{2}\Gamma_\beta\right) \\ &= \frac{1}{2}\pi_\alpha g^{\alpha\beta}\pi_\beta + V_{G/H}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} H_{\mathrm{orb}} &= \frac{1}{2} p_i g^{ij} p_j \\ &= \frac{1}{2}\left(p_\mu - \frac{\mathrm{i}\hbar}{2}\Gamma_\mu\right) g^{\mu\nu}\left(p_\nu + \frac{\mathrm{i}\hbar}{2}\Gamma_\nu\right) \\ &= \frac{1}{2} p_\mu g^{\mu\nu} p_\nu + V_{\mathrm{orb}}. \end{aligned} \tag{5.5}$$

The objects $\Gamma_\alpha = \Gamma^\beta{}_{\alpha\beta}$ and $\Gamma_\mu = \Gamma^\nu{}_{\mu\nu}$ are defined analogously to Γ_M. The "quantum potentials" $V_{G/H}$ and V_{orb} have the form

$$\begin{aligned} V_{G/H} &= \frac{\hbar^2}{4}\left(\partial_\alpha \Gamma^\alpha + \frac{1}{2}\Gamma^\alpha \Gamma_\alpha\right), \\ V_{\mathrm{orb}} &= \frac{\hbar^2}{4}\left(\partial_\mu \Gamma^\mu + \frac{1}{2}\Gamma^\mu \Gamma_\mu\right). \end{aligned} \tag{5.6}$$

The Heisenberg equations of motion describing the dynamics in G can be derived by means of the $(m+n)$-decomposition of the metric

$$\dot{p}_M = \frac{1}{\mathrm{i}\hbar}\left[p_M, H_G\right]. \tag{5.7}$$

Performing the direct calculation and using the commutative relations, one can find

$$\dot{\pi}_\alpha = \frac{1}{2}\pi_\beta \partial_\alpha g^{\beta\gamma}\pi_\gamma + \left(F_{\alpha\beta}\circ g^{\beta\gamma}\right)\circ \pi_\gamma + \partial_\alpha V_{G/H}, \tag{5.8}$$

$$\dot{p}_i = 0, \tag{5.9}$$

where we denote $F_{\alpha\beta} := F^i{}_{\alpha\beta}\circ p_i$.

So we can conclude from (5.8) and (5.9) that the motion of a particle is governed by a Lorentz-type force represented by the second term in the right hand side of (5.8). This object is determined by the strength tensor (4.15) of the gauge potential A^i_α. The motion of a particle on the orbit gH is completely flat due to the conservation law (5.9).

The main result of this section consists in the emergence of a gauge structure in quantum theory on a homogeneous manifold. Such a structure is induced by the additional degrees of freedom caused by an isotropy group. This result is not surprising, because the given theory can be considered as a version of the Kaluza–Klein scheme, which has been exhaustively investigated in a great number of works [3].

6 Coordinate representation and physical sector of states

The procedure for constructing the quantum space of states for quantum mechanics on a homogeneous Riemannian manifold with the simply transitive action of the transformation group G has been introduced in [2]. The results obtained are quite applicable in the case we are considering. The problem arising here is how to eliminate the unphysical states (this does not refer to quantum mechanics on G/H) from the whole set of states of the quantum mechanics on G. The simplest way to perform such a procedure consists of using the $(m+n)$-decomposition.

According to [2] the coordinate representation of the operators, corresponding to quantum mechanics on G, is defined by its action on the wave functions

$$\psi(x) := \langle x \,|\psi\rangle \tag{6.1}$$

(here $|x\rangle$ is an eigenvector of the coordinate operator and $|\psi\rangle$ denotes an arbitrary state vector) has the following form:

$$\hat{x}^M = x^M,$$
$$\hat{p}_M = -\mathrm{i}\hbar\left(\partial_M + \frac{1}{2}\Gamma_M\right). \tag{6.2}$$

Similarly we can write

$$\hat{H}_G = -\frac{\hbar^2}{2}\left(\partial_M + \Gamma_M\right)\eta^{MN}\left(\partial_M + \Gamma_M\right)$$
$$= -\frac{\hbar^2}{2}\frac{1}{\sqrt{\eta}}\partial_M\left(\sqrt{\eta}\eta^{MN}\partial_N\right), \tag{6.3}$$

where $\eta = \det(\eta_{MN})$.

The wave function (6.1) satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2}\frac{1}{\sqrt{\eta}}\partial_M\left(\sqrt{\eta}\eta^{MN}\partial_N\psi\right) = E\psi. \tag{6.4}$$

The coordinate representation of the generator of permissible variations on G has the form

$$\hat{G} = \varepsilon^A \hat{v}^M_A \circ \hat{p}_M = -\mathrm{i}\hbar\varepsilon^A v^M_A \partial_M, \tag{6.5}$$

which can be derived using (6.2) and the properties of the Killing vectors $\{v^M_A\partial_M : a = \overline{1,p}\}$.

Hence, the coordinate and momentum operators can be rewritten in terms of the $(m+n)$-decomposition as

$$\hat{x}^\mu = x^\mu, \quad \hat{p}_\mu = -\mathrm{i}\hbar\left(\partial_\mu + \frac{1}{2}\Gamma_\mu\right) \tag{6.6}$$

for the operators describing quantum mechanics on the orbit gH, and

$$\hat{x}^\alpha = x^\alpha, \quad \hat{p}_\alpha = -\mathrm{i}\hbar\left(\partial_\alpha + \frac{1}{2}\Gamma_\alpha\right) \tag{6.7}$$

for the operators describing quantum mechanics on the quotient space G/H.

Further we consider the procedure of the selection of the physical sector, or, in other words, the states represented by the subset $L_2(G/H) \subset L_2(G)$ that describes quantum mechanics on G/H.

The wave function $\psi \in L_2(G)$ performs the map

$$\psi : G \longrightarrow \mathbf{C}^n,$$

which can be restricted to the function on G/H by means of the section of $G(G/H, H)$:

$$s : G/H \longrightarrow G,$$
$$[g] \longrightarrow s([g]) \in gH \tag{6.8}$$

that meets the condition $\rho(s([g])) = g$ and performs the correspondence between the equivalence class $[g]$ and its representative $gh = s([g]) \in gH \subset G$ for some $h \in H$. In this expression the element $h \in H$ completely determines the section s; therefore we denote this section as s_h.

Hence

$$\psi \circ s_h := \phi : G/H \longrightarrow \mathbf{C}^n, \tag{6.9}$$
$$[g] \longrightarrow \phi([g]) = \psi(s_h([g])) \equiv \psi(gh)$$

is the wave function on G.

The matrix elements of the physical observables calculated on the wave functions $\phi = \psi \circ s \in L_2(G/H)$ have to be independent of the choice of the section s_h. This is possible if and only if the wave functions $\phi := \psi \circ s_h$ and $\phi' := \psi \circ s_{h'}$ are connected by the unitary transformation

$$\begin{aligned}\phi'([g]) \equiv \psi(gh') \equiv \psi(ghh^{-1}h') \\ = U(h,h')\psi(gh) = U(h,h')\phi([g]).\end{aligned} \quad (6.10)$$

Therefore one can show that the wave functions of the physical sector obey the condition

$$\psi_{\rm phys}(gh) = \sigma_{h^{-1}}\psi_{\rm phys}(g), \quad (6.11)$$

where $\sigma_{h^{-1}}$ is the right unitary representation of $H \subset G$ in $\mathbf{C}^n$ (while σ_h is the left one).

The representation of H on the physical states induces the representation of Lie(H) as

$$\tilde{\sigma}_i := \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \sigma_{h_i(\tau)} \in \mathrm{vect}(\mathbf{C}^n) \cong \mathbf{C}^n, \quad (6.12)$$

where $h_i(\tau) \in H$ is an integral curve for the basic element $T_i|_e \in T_e H$. The connection between (6.12) and the generator of coordinate transformation can be expressed by

$$\begin{aligned}\tilde{\sigma}_i\psi_{\rm phys} &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \psi_{\rm phys}(gh_i^{-1}(\tau)) \\ &= \left.\frac{\partial\psi_{\rm phys}(gh_i^{-1})}{\partial x^\alpha(gh_i^{-1})}\frac{\partial x^\alpha(gh_i^{-1})}{\partial x^\mu(h_i^{-1})}\ \frac{\mathrm{d}x^\mu(h_i^{-1})}{\mathrm{d}\tau}\right|_{\tau=0} \\ &= -L_i^\alpha(g)\partial_\alpha\psi_{\rm phys}.\end{aligned}$$

Hence, the wave functions of the physical sector of states satisfy the equation

$$L_i^\alpha\partial_\alpha\psi_{\rm phys} \equiv \frac{1}{\mathrm{i}\hbar}\hat{p}_i\psi_{\rm phys} = -\tilde{\sigma}_i\psi_{\rm phys}. \quad (6.13)$$

This points to the fact that $p_i = L_i^\mu \circ p_\mu$ describes the representation of Lie(H).

Using (6.13) one can express the "horizontal derivative" D_α in terms of the generators $\tilde{\sigma}_i$. The derivative D_α acts on the physical sector by

$$D_\alpha = \left(\partial_\alpha + A_\alpha^i\tilde{\sigma}_i\right)\psi_{\rm phys}. \quad (6.14)$$

This formula coincides with the definition of the invariant derivative associated with the action of the gauge group.

Finally, using (6.14) in (6.3) we find a coordinate representation of $\hat{H}_{G/H}$ on the physical sector:

$$\hat{H}_{G/H} = -\frac{\hbar^2}{2}\left(D_\alpha + \Gamma_\alpha\right)g^{\alpha\beta}D_\beta - \frac{\hbar^2}{2}\hat{C}, \quad (6.15)$$

where $\hat{C} = \eta^{ij}\tilde{\sigma}_i\tilde{\sigma}_j$ is the Casimir operator of the unitary representation of H.

According to the general theory of unitary representations [7], the irreducible representations of Lie groups are finite dimensional and can be described by eigenvalues of the Casimir operator.

Therefore, the given irreducible unitary representation describes one from several inequivalent theories on G/H based on the Hamiltonian (6.3).

The analysis of inequivalent quantum theories has been performed in [4] in terms of the representation theory Weyl relations. Our final results, obtained by means of the extended Schwinger quantization scheme, completely correspond to the results of ([4]).

7 Summary and discussion

Quantum mechanics on the homogeneous manifold G/H has been constructed using our extension of the Schwinger quantization procedure. The essential feature of quantum mechanics on G/H consists of the appearance of the gauge structure induced by some unitary (irreducible) representation of the isotropy subgroup $H \subset G$ (H plays the role of a gauge group). The gauge field corresponds to the connection 1-form of the fiber bundle $G(G/H, H)$. There exist a number of inequivalent quantum theories classified by eigenvalues of the Casimir operator of the unitary representation.

A successful development of quantum mechanics on a homogeneous Riemannian manifold with simply and non-simply transitive transformation groups of isometries shows that the extended Schwinger quantization scheme is suitable for constructing quantum mechanics on a manifold with a group structure. This approach can be applied to analyze a number of models such as Kaluza–Klein theories [3] or to generalize the simplest hadron models [6].

Appendix

A Realizations of Lie groups on manifolds

Lie groups and Lie algebras

Let G be a p-dimensional Lie group with a local coordinate system $\{x^M(g) : g \in G, M = \overline{1,p}\}$ at a point $g \in G$.

Left and right translations are defined by

$$\begin{aligned} L_g : G &\longrightarrow G, & R_g : G &\longrightarrow G, \\ h &\longrightarrow L_g h := gh, & h &\longrightarrow R_g h := hg. \end{aligned} \quad (A.1)$$

In the tangent space T_hG, L_g and R_g induce the following differential maps

$$\mathrm{d}L_g : T_hG \longrightarrow T_{gh}G, \quad \mathrm{d}R_g : T_hG \longrightarrow T_{hg}G. \quad (A.2)$$

In the local coordinate system $\{x^M(\cdot)\}$ the element $A|_h \in T_hG$ can be written as

$$A|_h = a^M(h)\,\partial_M|_h\,. \quad (A.3)$$

Therefore we can express the transformations (A.2) in the local form

$$\mathrm{d}L_g(A|_h) = a^M(h)\frac{\partial x^N(gh)}{\partial x^M(h)}\frac{\partial}{\partial x^N(gh)}$$

$$= a^M(h) L_M^N(gh,h) \frac{\partial}{\partial x^N(gh)}, \quad (A.4)$$

$$\mathrm{d}R_g(A|_h) = a^M(h) \frac{\partial x^N(hg)}{\partial x^M(h)} \frac{\partial}{\partial x^N(hg)}$$
$$= a^M(h) R_M^N(hg,h) \frac{\partial}{\partial x^N(hg)}, \quad (A.5)$$

where we denote the matrices of the left and right translations together with their inverses by

$$L_M^N(gh,h) = \frac{\partial x^N(gh)}{\partial x^M(h)}, \quad \overline{L}_N^M(h,gh) = \frac{\partial x^M(h)}{\partial x^N(gh)}, \quad (A.6)$$

$$R_M^N(hg,h) = \frac{\partial x^N(hg)}{\partial x^M(h)}, \quad \overline{R}_N^M(h,hg) = \frac{\partial x^M(h)}{\partial x^N(hg)} \quad (A.7)$$

(here $\overline{L} = L^{-1}$, $\overline{R} = R^{-1}$).

The Lie algebras of left- and right-invariant vector fields are constructed by left and right translations of the elements of T_eG ($e \in G$ denotes the unit element) respectively.

We denote the basic element of T_eG by

$$T_A|_e = \frac{\partial}{\partial x^A(e)} := \partial_A|_e. \quad (A.8)$$

Then the set of left- and right-invariant vector fields

$$L_A|_g := \mathrm{d}L_g \left(\left.\frac{\partial}{\partial x^A}\right|_e \right)$$
$$= \mathrm{d}L_g(T_A|_e) = L_A^M(g) \left.\frac{\partial}{\partial x^M}\right|_g, \quad (A.9)$$

$$R_A|_g := \mathrm{d}R_g \left(\left.\frac{\partial}{\partial x^A}\right|_e \right)$$
$$= \mathrm{d}R_g(T_A|_e) = R_A^M(g) \left.\frac{\partial}{\partial x^M}\right|_g \quad (A.10)$$

form a basis of the left [right] Lie algebra Lie(G) of the Lie group G. The corresponding matrices $L(g) = \{L_A^M(g)\}$ and $R(g) = \{R_A^M(g)\}$ are obtained by the reduction of (A.6) and (A.7):

$$L_A^M(g) = L_A^M(gh,h)|_{h=e},$$
$$\overline{L}_M^A(g) = \overline{L}_M^A(h,gh)|_{h=e}, \quad (A.11)$$

$$R_A^M(g) = R_A^M(hg,h)|_{h=e}$$
$$\overline{R}_M^A(g) = \overline{L}_M^A(h,hg)|_{h=e}. \quad (A.12)$$

Here, as in Sect. 2 the first lot of Latin indices $A, B, \ldots = \overline{1,p}$ is used to indicate the group degrees of freedom (i.e. the frame of T_eG), and the final one $M, N, \ldots = \overline{1,p}$ describes the tensor degrees of freedom.

The structure equations for the left Lie algebra are

$$[L_A, L_B] = c^C{}_{AB} L_C, \quad (A.13)$$

or, in a local form

$$L_A^M \partial_M L_B^N - L_B^M \partial_M L_A^N = c^C{}_{AB} L_C^N. \quad (A.14)$$

The right Lie algebra has the basis $\{R_A = R_A^M \partial_M\}$ and obeys similar structure equations; these can be obtained by the following replacement

$$L_A^M \longrightarrow R_A^M, \quad c^C{}_{AB} \longrightarrow -c^C{}_{AB}.$$

The inverse matrix $\overline{L}$ satisfies the Maurer–Cartan equations

$$\partial_M \overline{L}_N^A - \partial_N \overline{L}_M^A = -c^A{}_{BC} \overline{L}_M^B \overline{L}_N^C. \quad (A.15)$$

Note that left and right Lie algebras are commutative,

$$R_A^M \partial_M L_B^N = L_B^M \partial_M R_A^N. \quad (A.16)$$

Action of Lie group on manifold

Let M and G be a smooth manifold and the Lie group of right transformations on M defined by the map

$$\rho : M \times G \longrightarrow M, \qquad (x,g) \longrightarrow y := \rho_g x, \quad (A.17)$$

with the following properties

(1) $\forall g_{1,2} \in G, \quad \rho_{g_1} \circ \rho_{g_2} = \rho_{g_2 g_1}$;
(2) $\rho_e = \mathrm{id}_M$ (the identity map);
(3) $\forall g \in G, \quad \rho_g : M \to M$ is a diffeomorphism ($\rho_g \in \mathrm{Diff}(M)$).

The realization (A.17) of the Lie group as the transformation group induces the realization of the Lie algebra by the following procedure.

Let $A \in \mathrm{Lie}(G)$, $g_A(\tau) := \exp(\tau A) \in G$, be an integral curve of A (one-parametric subgroup of G, $g_A(0) = e$). Then the vector

$$\tilde{\rho}_A|_x = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \rho_{\exp(\tau A)}(x) \in T_xM$$

defines the realization of $A \in \mathrm{Lie}(G)$. Due to the homomorphic nature of the map $\rho : G \to \mathrm{Diff}(M)$, the vector subspace $\tilde{\rho}_{\mathrm{Lie}(G)} \subset \mathrm{vect}(M)$ is finite dimensional (vect(M) denotes the space of vector fields over M).

A Lie group can be realized on itself with the use of left [right] translations. The induced realization of the Lie algebra coincides with the Lie algebra of right- [left-] invariant vector fields. The basic element of T_eG

$$T_A|_e = \delta_A^M \partial_M|_e$$

is represented by the vector

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} x^M(g g_A(\tau)) \partial_M|_g = L_A^M(g) \partial_M|_g. \quad (A.18)$$

Similarly, for the left action we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} x^M(g_A(\tau) g) \partial_M|_g = R_A^M(g) \partial_M|_g. \quad (A.19)$$

Metric of Lie group

The tangent space T_eG at the unit element $e \in G$ is a p-dimensional vector space. We assume that there exists a scalar product defined by

$$\begin{aligned}(\cdot,\cdot) : T_eG \times T_eG &\longrightarrow \mathbf{R},\\ \langle A|_e, B|_e\rangle &\longrightarrow (A|_e, B|_e).\end{aligned} \tag{A.20}$$

The scalar products of the basic elements $\{T_A|_e : A = \overline{1,p}\}$ form the matrix

$$\eta_{AB} := (T_A|_e, T_B|_e) = \text{const} \tag{A.21}$$

that obeys the tensor transformation law under Gl_p transformations in T_eG.

Using (A.21) one can easily show that the scalar product of left-invariant fields $\{L_A\}$ on G equals the matrix (A.21):

$$(L_A|_g, L_B|_g) = (L_A|_e, L_B|_e) \equiv (T_A|_e, T_B|_e) = \eta_{AB}. \tag{A.22}$$

This scalar product can be identified with the left-invariant metric of G. If G admits a subgroup $H \subset G$, the metric can be chosen in diagonal form:

$$\{\eta_{AB}\} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ij} \end{pmatrix}, \quad \{\eta^{AB}\} = \begin{pmatrix} g^{ab} & 0 \\ 0 & g^{ij} \end{pmatrix}, \tag{A.23}$$

by means of a Gl_p transformation. The tensor $\{g_{ij}\}$ in (A.23) has the meaning of the metric of T_eH.

The components of the metric of G can be defined with respect to the holonomic frame of vect(G) by the following scalar product:

$$\eta_{MN}(g) := \left(\left.\frac{\partial}{\partial x^M}\right|_g, \left.\frac{\partial}{\partial x^N}\right|_g \right). \tag{A.24}$$

Thus, using (A.22) and (A.24) we can write

$$\eta_{AB} = (L_A|_g, L_B|_g) = L_A^M(g) L_B^N(g) \left(\left.\frac{\partial}{\partial x^M}\right|_g, \left.\frac{\partial}{\partial x^N}\right|_g \right).$$

Therefore

$$\eta_{MN} = \overline{L}_M^A(g) \overline{L}_N^B(g) \eta_{AB} \tag{A.25}$$

is the left-invariant metric of G.

If the coordinate transformation is performed by the right translations, the metric (A.25) transforms as

$$\eta_{M_1M_2}(g_1g_2) = \frac{\partial x^{N_1}(g_2)}{\partial x^{M_1}(g_1g_2)} \frac{\partial x^{N_2}(g_2)}{\partial x^{M_2}(g_1g_2)} \eta_{N_1N_2}(g_2). \tag{A.26}$$

Such a transformation law means that the right translations on G are isometric transformations of the metric (A.25). Equivalently, the Lie variation of (A.25) associated with the right-invariant vector field vanishes, i.e. the right Lie algebra consists of Killing vectors.

However, the left-invariant vector field is not a Killing vector of (A.25) in the general case. This is possible if and only if the Lie group is semisimple. In this case $\{\eta_{AB}\}$ coincides with an adjoint-invariant Killing–Cartan form and the right-invariant metric of G is the same as the left-invariant one. Therefore, the Killing vectors of $\{\eta_{MN}\}$ correspond to both Lie algebras.

B Connection 1-form of principal fiber bundle

The connection 1-form $\omega \in \mathcal{A}^1(G, \mathrm{Lie}(H))$ (see, for example, [5]) of the principal fiber bundle $G(G/H, H)$ can be defined as the restriction of the Maurer–Cartan form of G to H by the identification of Lie(H) with T_eH:

$$\omega : \mathrm{vect}(G) \longrightarrow \mathrm{Lie}(H) \cong T_eH. \tag{B.1}$$

In this case we can write

$$\omega|_g = \omega_M(g) \mathrm{d}x^M|_g, \quad \omega_M(g) = \overline{L}_M^i T_i|_e, \tag{B.2}$$

where $\{T_i|_e : i = \overline{n+1,p}\}$ denotes a basis of T_eH.

In the local coordinate system corresponding to the $(m+n)$-decomposition (where $L_i^\alpha = 0$), the connection 1-form has the following form:

$$\begin{aligned}\omega|_G &= \omega_M \mathrm{d}x^M|_g = \overline{L}_M^i L_i^N \mathrm{d}x^M|_g \otimes \left.\frac{\partial}{\partial x^N}\right|_g \\ &= \omega_M^\mu \mathrm{d}x^M|_g \otimes \left.\frac{\partial}{\partial x^\mu}\right|_g.\end{aligned} \tag{B.3}$$

Due to the properties of the matrices L and $\overline{L}$ in such a coordinate system one can observe that

$$\begin{aligned}\omega^\mu &= \omega_M^\mu \mathrm{d}x^M \\ &= \mathrm{d}x^\mu + \overline{L}_\alpha^i L_i^\mu \mathrm{d}x^\alpha = \mathrm{d}x^\mu + A_\alpha^\mu \mathrm{d}x^\alpha,\end{aligned} \tag{B.4}$$

where $A_i^\mu = \overline{L}_\alpha^i L_i^\mu$ has the meaning of a gauge field.

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